EQUILIBRIUM STATES OF AN ELECTRON BEAM IN A MAGNETIC FIELD

V. A. Malafaev and D. N. Novichkov

UDC 621.384.6

There has recently been an appreciable increase of interest in the study of equilibrium states of electron beams which are decompensated or partially compensated by ions, and confined by a magnetic field. In addition to the programs listed in [1] which initiated this interest, there are problems of the adiabatic acceleration of electron and ion beams [2], the recovery of their energy, and compression. Fotin et al. [3] treated one aspect of the analysis of decompensated electron beams by considering their possible application to the problem of the transport of electrical energy on a commercial scale by using electron beams. The use of decompensated beams opens up the possibility of transporting electromagnetic energy in the form of the Poynting flux, as in an ordinary coaxial cable. In this case the ratio of the kinetic and potential energies is of definite concern in the analysis of states. The limiting currents of charged beams in an infinitely strong magnetic field were investigated in [4-7]. Zharinov et al. [6] analyzed states and drew certain conclusions about the transformation of energy from electromagnetic to kinetic and back. They showed that a decompensated electron beam can be decelerated or accelerated in an equipotential beam line by a suitable choice of beam profile, determined by the external magnetic field. In a magnetic field of finite strength electrons have an additional degree of freedom which permits their rotation about the magnetic lines of force. It is of interest to analyze equilibrium states and energy diagrams of charged beams in a magnetic field, taking account of the transformation of electromagnetic energy into translational and rotational kinetic energy. We consider an electron beam injected from a cathode of radius r_K and accelerated across a potential difference φ_K in a magnetic field H_K. The total potential difference between the cathode and the beam line is φ_R. The beam is ejected into a beam line of radius R with an accompanying magnetic field H. We analyze equilibrium states at large distances from the point of injection or for an adiabatically slow variation of H along the length of the beam. We do not consider the stability of the equilibrium states.

Nonrelativistic Case. The equation of motion of an edge electron in the drift region has the form

$$\ddot{r}_0 = \eta E_r + \omega^2 r_0 - \omega\omega_c r_0, \tag{1}$$

where r_0 is the radius of the beam, E_r is the radial electric field of the beam, $\eta = e/m$ is the specific charge of the electron, $\omega_c = \eta H/c$ is the cyclotron frequency, ω is the angular velocity given by the Bush theorem

$$\omega = (\omega_c/2)\left(1 - \alpha r_K^2/r_0^2\right), \tag{2}$$

where $\alpha = H_K/H$, and r_K is the radius of the beam at the cathode.

It follows from Eq. (2) that for $\alpha r_K^2/r_0^2 \ll 1$ the angular velocity of the electrons is determined solely by the magnetic field intensity H. In this case Eq. (1) describes Brillouin flow in which the longitudinal velocity and the electron density are constant along the radius [8]. For $\alpha r_K^2/r_0^2 \leqslant 1$ the rotational energy approaches zero. In this case the longitudinal velocity along the axis is somewhat lower at the edge because of the potential drop on the beam. This drop can be neglected when $2\ln(R/r_0) \gg 1$. Satisfying at least one of the inequalities $\alpha r_K^2/r_0^2 \ll 1$ or $2\ln(R/r_0) \gg 1$ is equivalent to satisfying the inequality

$$r_0^2/r_K^2 \gg \alpha/\ln\left(R^2/\alpha r_K^2\right) \tag{3}$$

We consider beams for which Eq. (3) is valid, i.e. beams in which the density and longitudinal velocity of the electrons are constant along a radius. This condition is satisfied either by sufficiently narrow beams or beams having an appreciable rotational energy. From Gauss's theorem the radial electric field $E_r = 2I/v_z r_0$, where I is the beam current and v_z is the longitudinal velocity of the electrons. Substituting the value of the radial field into (1), and using (2), we obtain the following expression for the equilibrium radius of

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 24-30, September-October, 1983. Original article submitted July 22, 1982.

0021-8944/83/2405-0627$07.50

the beam by setting $\ddot{r}_0 = 0$:

$$x^4 - \frac{2\omega_K^2}{\omega_c^2} x^2 - \alpha^2 = 0, \tag{4}$$

where

$$x = r_0/r_K; \quad \omega_K^2 = 4I\eta/v_z r_K^2. \tag{4'}$$

We note that the rotation of electrons gives rise to a diamagnetic field which affects the rotational energy. The diamagnetism is small if

$$(Ie/(v_z mc^2))(1 - \alpha r_K^2/r_0^2) \ll 1. \tag{5}$$

Taking account of the space charge limitation of the current, condition (5) is always satisfied in the nonrelativistic case.

For a constant longitudinal velocity and density of the electrons along a radius the law of conservation of the total energy flux through a cross section of the beam line has the form

$$\frac{I}{2\eta}(v_z^2 + \widehat{v}_\theta^2) + \frac{c}{4\pi}\int_0^R E_r H_\theta 2\pi r \, dr = I\varphi_R, \tag{6}$$

where H_θ is the component of the self-magnetic field of the beam. According to (2), the rotational velocity averaged over the cross section is

$$\widehat{v}_\theta^2 = (\omega_c^2 r_0^2/8)(1 - \alpha r_K^2/r_0^2)^2. \tag{7}$$

The first term in (6) corresponds to the transport of kinetic energy by the beam, and the second term to the transport of electromagnetic energy. Performing the integration in (6), and transforming by using (4'), we obtain

$$2\eta\varphi_R - (v_z^2 + \widehat{v}_\theta^2) = (\omega_K^2 r_K^2/4)(1 + 4\ln R/xr_K). \tag{8}$$

Equation (8) is valid only for nonpulsed beams, since the component v_r has been eliminated. Using (4), (7), and (8), we obtain the dependence of the equilibrium radius x on I, φ_R, R, r_K, and α in the form

$$i^2 = \frac{I^2}{8\eta\varphi_R^3} = B^2\frac{(x^4-\alpha^2)^2}{x^4}\left[1 - B\frac{(x^2-\alpha)^2 + (x^4-\alpha^2)\left(1 + 4\ln\frac{R}{xr_k}\right)}{x^2}\right], \tag{9}$$

where $B = \omega_c^2 r_K^2/(16\eta\varphi_R)$.

Equation (9) is illustrated graphically in Fig. 1 for $R/r_K = 10$, $\alpha = 0$, and $B = 1.63 \times 10^{-2}$, 8.4×10^{-3}, 5×10^{-3}, 4.8×10^{-3}, and 3.3×10^{-3} for curves 1-5 respectively. Analysis of Eq. (9) for $x^2 \gg \alpha$ shows that for $B \geqslant r^2/2R_K^2$, which is equivalent to $R\omega_c/2 \geqslant \sqrt{2\eta\varphi_R}$ there are two values of the equilibrium radius for a given current. This case corresponds to curves 1-3 of Fig. 1.

When the current is increased and B is fixed, one of the equilibrium radii is increased, and the other is decreased. At the maximum current i_{max}^B the two radii coincide. If $B \leqslant (r_K^2/3R^2)e^{1/3}$ or $R\omega_c/2e^{1/3} \leqslant \sqrt{2\eta\varphi_R/3}$, the equilibrium radius increases monotonically with increasing beam current, as shown by curve 5. Finally, in the range $r_K^2/2R^2 > B > (r_K^2/3R^2)e^{1/3}$ for $i < 1/9$ there are three values of the equilibrium radius (curve 4).

The extreme values of the currents for a fixed B and an arbitrary α are related to the equilibrium radius by the expression

$$i_e^B = 2\left(\frac{2\alpha^2}{x^4+\alpha^2} + 4\ln\frac{R}{xr_K}\right)^{1/2}\left[\frac{x^2-\alpha}{(x^2+\alpha)(x^4+\alpha^2)} + 3\left(1 + 4\ln\frac{R}{xr_K}\right)\right]^{-3/2}. \tag{10}$$

For $x^2 \gg \alpha$ and $x \leqslant (R/r_K)e^{-1/6}$, Eq. (10) determines the value of i_{max}^B, and for $x > (R/r_K)e^{-1/6}$ it determines the value of i_{min}^B. The parameter B is related to x by the expression

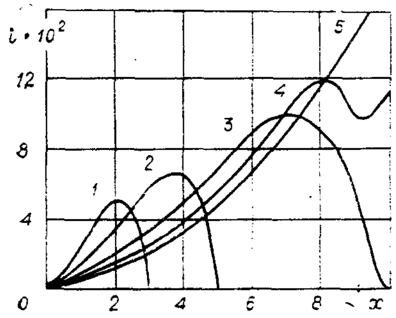

Fig. 1

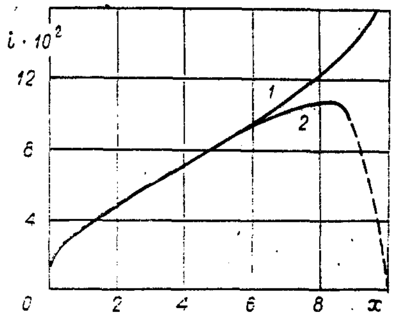

Fig. 2

$$B = \frac{2x^2}{x^4 - \alpha^2}\left[\frac{(x^2-\alpha)^3}{(x^2-\alpha)(x^4+\alpha^2)} + 3\left(1 + 4\ln\frac{R}{xr_K}\right)\right]^{-1}.$$

The maximum value of the current i^x_{max} for a fixed radius, determined from the condition $di/dB = 0$, is

$$i^x_{max} = \frac{2}{3\sqrt{3}}(x^2+\alpha)\left[(x^2-\alpha) + (x^2+\alpha)\left(1 + 4\ln\frac{R}{xr_K}\right)\right]^{-1}$$

at

$$B = \frac{2}{3}x^2\left[(x^2-\alpha)^2 + (x^4-\alpha^2)\left(1 + 4\ln\frac{R}{xr_K}\right)\right]^{-1}. \tag{11}$$

Curves 1 and 2 in Fig. 2 are plots of i^x_{max} and i^B_e for $\alpha = 0$. The maximum value of i^B_e is reached at $x = (R/r_K)e^{-1/6}$. At small x the two curves practically coincide, but for $x \geqslant (R/r_K)e^{-1/6}$ they differ appreciably from one another. The right-hand branch of the i^B_e curve, shown dashed, corresponds to the values of i^B_{min}.

We turn now to an analysis of the energy state of the electron beam. Substituting (11) into (7), we obtain the value of the rotational energy for currents equal to i^x_{max}:

$$\varphi_\theta = \frac{2}{3}\varphi_R(x^2-\alpha)\left[(x^2-\alpha) + (x^2+\alpha)\left(1 + 4\ln\frac{R}{xr_K}\right)\right]^{-1}, \tag{12}$$

where $\varphi_\theta = \hat{v}_\theta^2/2\eta$. According to (12) the rotational energy approaches zero as $x^2 \to \alpha$. It follows from (11) that this is possible for $B \to \infty$. The maximum value of φ_θ is reached at $x = R/r_K$. If $R/r_K \gg \alpha$, the maximum rotational energy is $\varphi_R/3$.

An important characteristic of the state of an electron beam is the ratio of the kinetic and potential energies of the beam. From (3) and (8)

$$\frac{\varphi_K}{\varphi_R} = 1 - \left[B\frac{(x^4-\alpha^2)}{x^2}\right]\left(1 + 4\ln\frac{R}{xr_K}\right), \tag{13}$$

where $\varphi_K = (v_z^2 + \hat{v}_\theta^2)/2\eta$, which determines the ratio of the kinetic to the total energy as a function of the equilibrium radius of the beam. In regimes of the type 1-3 (Fig. 1) where two values of the equilibrium radius are possible, Eq. (13) shows that the larger of these corresponds to the smaller kinetic energy.

The state diagrams relating the beam current to φ_K/φ_R, can be obtained from Eqs. (9) and (13) by eliminating x. These equations cannot be solved analytically in general form. Curves 1-4 of Fig. 3a ahow a graphical solution of these equations for $B = 3.3 \times 10^{-3}$, 5×10^{-3}, 8.4×10^{-3}, and 1.63×10^{-2} respectively. It should be noted that each value of the current along a curve corresponds to a definite value of the equilibrium radius, which varies from point to point in accordance with Eq. (9). The ratio φ_K/φ_R for the same B is determined by the geometry of the system, i.e., by r_0/r_K and R/r_K. For example, for an electron beam with a current $i = 0.044$ propagating along a slowly varying magnetic field, the equilibrium radius changes with a change in the kinetic energy. Depending on the initial conditions, the kinetic energy of the beam may increase and the equilibrium radius decreases (the upper left-

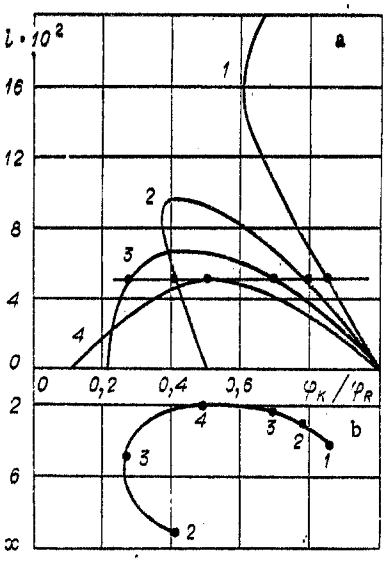

Fig. 3

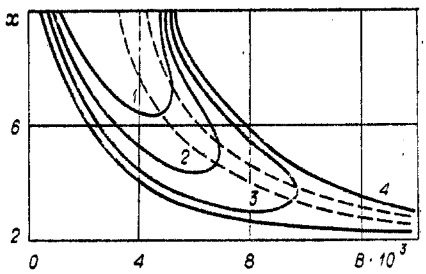

Fig. 4

hand branch of the curve in Fig. 3b with B = 5×10^{-3}, 8.4×10^{-3}, 1.64×10^{-2}, 8.4×10^{-3}, 5×10^{-3}, and 3.3×10^{-3} at points 2, 3, 4, 3, 2, and 1 respectively) or the kinetic energy and the beam radius may both increase (right-hand branch). Since the kinetic energy of the longitudinal motion $\varphi_{\parallel} = v_z^2/2\eta$ is

$$\frac{\varphi_{\parallel}}{\varphi_R} = 1 - \left[\frac{4B(x^4-\alpha^2)}{x^2}\right]\ln\left(\frac{R}{xr_{\text{к}}}\right) - 2B(x^2-\alpha),$$

on the upper left-hand branch the increase in kinetic energy is the result of an increase in the longitudinal velocity of the electrons, while on the right hand branch it is the result of an increase in the rotational energy. The lower left-hand branch of the curve shows that an increase in the kinetic energy is accompanied by an increase in the radius and a substantial decrease in the longitudinal velocity of the electrons.

The shape of the beam in an adiabatically slowly varying magnetic field ($\dot{\omega}_c \ll \omega_c^2$) can be obtained by solving Eq. (9) for x. Curves 1-4 of Fig. 4 show the dependence of x on B for $\alpha = 0$ and $i = 10^{-1}$, 8×10^{-2}, 6×10^{-2}, and 4×10^{-2} respectively. The value of dx/dB for $\alpha = 0$ is

$$\frac{dx}{dB} = \frac{x}{2B}\left[1 - 3Bx^2\left(2\ln\frac{R}{xr_{\text{к}}} + 1\right)\right]\left[1 - 2Bx^2\left(3\ln\frac{R}{xr_{\text{к}}} + 1\right)\right]^{-1}.$$

The functions $B = \left[3x^2\left(2\ln\frac{R}{xr_{\text{к}}} + 1\right)\right]^{-1}$ and $B = \left[2x^2\left(3\ln\frac{R}{xr_{\text{к}}} + 1\right)\right]^{-1}$ are shown by the open curves of Fig. 4. If a beam with a current $i < 1/9$ is injected into a beam line with $B < (r_{\text{к}}^2/3R^2)\,e^{1/3}$, then for a magnetic field which increases slowly along the length, the beam must reach the state with the smaller radius. For a specific field the minimum value of the radius is found from the condition dx/dB = 0. After this the radius increases somewhat. The maximum value of B for which an equilibrium state of the beam still exists is determined from the condition dx/dB = ∞. At this point the two equilibrium radii coincide. If the field is decreased, the beam may make a transition to the state with the larger radius, or return to the state with the smaller radius. If the magnetic field is increased when $i \geqslant 1/9$, the radius of the beam at first decreases, and then increases. The minimum value of the radius is determined from the condition dx/dB = 0. The behavior of the beam for finite α does not differ quantitatively from the behavior for $\alpha = 0$. In this case all the curves are displaced along the x axis by $\sqrt{\alpha}$.

Relativistic Case. By analogy with the case considered, the equilibrium condition in the relativistic case is written in the form

$$1 + B_c\frac{(x^2-\alpha)^2}{x^2} = \frac{B_c(x^4-\alpha^2)}{2i_A x^2}\left[\gamma^2 - 1 + B_c\frac{(x^2-\alpha)^2}{x^2}\right]^{1/2}, \tag{14}$$

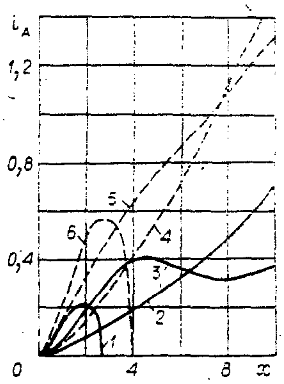

Fig. 5

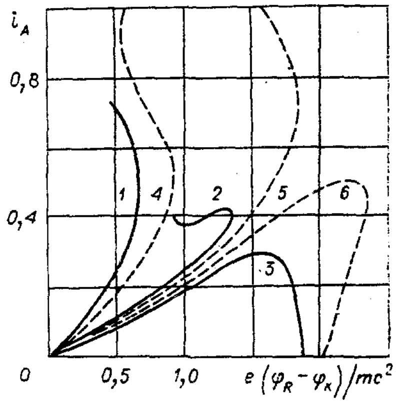

Fig. 6

where $B_c = \omega_c^2 r_K^2/4c^2$; $i_A = Ie/mc^3$; $\gamma = (1-\beta^2)^{-1/2}$; $\beta^2 = \beta_z^2 + \beta_\theta^2$.

Making the same assumptions as in the nonrelativistic case, we obtain for the law of conservation of the total energy flux

$$\gamma_R - \frac{2}{3}\frac{1}{(1-\beta_z^2)^{1/2}}\frac{x^2}{B_c(x^2-\alpha)^2}\left[\left(1+B_c\frac{x^2-\alpha}{x^2}\right)^{3/2}-1\right] = \frac{i_A}{2\beta_z}\left(1+4\ln\frac{R}{xr_K}\right),$$

where $\gamma_R = 1 + e\varphi_R/mc^2$; $e\varphi_R$ is the total energy of the electrons. Eliminating β_z from (14) and (15), we obtain

$$i_A = \frac{B_c(x^2-\alpha)^2}{2x^2\left[1+B_c\frac{(x^2-\alpha)^2}{x^2}\right]^{1/2}}[\varkappa^2-1]^{1/2},$$

where

$$\varkappa = \gamma_R\left\{\frac{2}{3}\frac{\left[1+B_c\frac{(x^2-\alpha)^2}{x^2}\right]^{3/2}-1}{B_c\frac{(x^2-\alpha)^2}{x^2}} + \frac{B_c(x^4-\alpha^2)}{4x^2\left[1+B_c\frac{(x^2-\alpha)^2}{x^2}\right]^{1/2}}\left(1+4\ln\frac{R}{xr_K}\right)\right\}^{-1}.$$

The solid and open curves of Fig. 5 show the dependence of i_A on x for γ_R = 3.1 and 4.1 respectively. The picture remains qualitatively the same as before — for sufficiently large magnetic fields (large B_c) there are two equilibrium radii of the beam. As B_c is increased, both radii increase, and in a certain range of B_c there are three values of the equilibrium radii (B_c = 0.27, 0.01, 0.09, 0.03, 0.09, and 0.27 for curves 1-6 respectively). If B_c is small enough, the beam radius increases monotonically with increasing current. The maximum value of the current is reached when the beam completely fills the beam line. Figure 6 shows the current as a function of the electromagnetic energy transported by the beam (B_c = 0.01, 0.09, 0.27, 0.01, 0.09, and 0.27 for curves 1-6 respectively) for the same values of γ_R. It is clear from Fig. 6 that to each radius there corresponds a definite electromagnetic energy whose magnitude depends rather strongly on the applied magnetic field. It should be noted that in the relativistic case the condition for small diamagnetism is satisfied only for current $i_A \ll \beta\gamma$ or for sufficiently narrow beams — $2\ln(R/xR_K) \gg 1$. Therefore, the quantitative results in the relativistic case are approximate.

In conclusion we note the following:

1. Our analysis of the equations of motion and the law of conservation of the total beam energy flux in a longitudinal magnetic field shows that for a fixed magnetic field there are maximum currents whose magnitude increases with a decrease of the field. This is related to an increase in beam radius and a transformation of part of the potential energy of the electrons into kinetic energy, which decreases the space charge density. In addition, for a fixed beam radius there are maximum currents which for small values of the radius coincide with the maximum currents for a fixed magnetic field, while for larger radii may substantially exceed them (Fig. 2).

2. The rotational energy of beam electrons increases with increasing current, and may reach an appreciable fraction of the total energy. The rotational energy decreases with an increase in the magnetic field, and approaches zero in the limit of an infinitely large field.

3. State diagrams relating the energy characteristics of the beam, the current, and the geometrical parameters enable one to distinguish three regions differing in the initial state. The transition from one region to another for an adiabatically slow variation of the magnetic field enables one to produce different transformations of the energy of the system, ensuring an adiabatic acceleration and deceleration, and also an appreciable compression of the beam within the framework of the model considered.

The authors thank A. V. Zharinov for posing the problem and for a discussion of the paper.

LITERATURE CITED

1. R. Davidson, Theory of Nonneutral Plasmas, Benjamin, New York (1974).
2. P. Sprangle, A. T. Drobot, and W. M. Manheimer, "Collective ion acceleration in converging guide," Phys. Rev. Lett., 36, 1180 (1976).
3. V. P. Fotin, V. V. Glebov, V. A. Malafaev, D. N. Novichkov, V. I. Perevodchikov, and A. V. Zharinov, "Simulation of phenomena associated with transport of decompensation and quasi-neutral electron beams for industrial scale transmission of electric power," in: Fourth Intern. Topical Conf. on High-Power Electron and Ion Beams, Palaiseau (1981).
4. L. P. Smith and P. L. Hartman, "The formation and maintenance of electron and ion beams," J. Appl. Phys., 11, 220 (1940).
5. L. É. Bakhrakh, "Limiting current of an electron beam under the assumption of a constant space charge density," Radiotekh. Elektron., 10, No. 6 (1965).
6. A. V. Zharinov, D. N. Novichkov, and A. S. Chikhachev, "Equations of state of an electron beam," Radiotekh., Elektron., 23, No. 8 (1978).
7. B. N. Breizman and D. D. Ryutov, "Powerful relativistic electron beams in a plasma and in a vacuum (theory)," Nucl. Fusion, 14, 873 (1974).
8. L. Brillouin, "A theorem of Larmor and its importance for electrons in magnetic fields," Phys. Rev., 67, 260 (1945).

NUMERICAL CALCULATIONS OF STATIONARY STATES OF MAGNETIC SELF-INSULATION OF VACUUM LINES

G. G. Golovin, A. V. Gordeev
B. D. Korolev, V. P. Smirnov,
and A. S. Chernenko

UDC 533.916 : 517.949.8

Magnetic self-insulation of vacuum gaps permits attaining electric fields of $> 10^6$ V/cm due to screening of the negative electrode by a layer of magnetized electrons [1]. As a result, it is possible to transmit energy fluxes along vacuum lines and to concentrate them to densities $\geqslant 10^{12}$ W/cm^2, which finds application, in particular, in large-scale systems, for example, Angara-5 [2]. In spite of the broad practical application of self-insulation, there is as yet no complete theory of the equilibrium of electron layers. The best developed models are the hydrodynamic Brillouin model and the kinetic model with one type of trajectory. The hydrodynamic model, which does not tale into account the pressure in the electron layer (Brillouin flow), describes well cylindrical lines. The more realistic kinetic model, which takes into account one type of electron trajectory, predicts the existence of equilibrium configurations only for flat and cylindrical lines and, in addition, in the latter, the external electrode must be negative [3]. The important case of converging conical lines, which is important for concentrating energy flux, is described only approximately by the hydrodynamic model. In the self-consistent kinetic as well as in the single-frequency approximations, there are no solutions, which is a result of the dependence of the azimuthal magnetic field on the distance to the apex of the cone [4]. Great diffi-

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 31-37, September-October, 1983. Original article submitted August 9, 1982.

 0021-8944/83/2405-0632$07.50